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# Heat transfer in the laminar flow of a non-Newtonian fluid in a porous annulus by the method of quasilinearization

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## 1. INTRODUCTION

IN THE last two decades, the problem of flow and heat transfer through an annulus with porous walls has attracted the attention of mathematicians and engineers; the problems have been studied due to their numerous applications to the cases of transpiration cooling, gaseous diffusion, oil field operations, boundary layer control, etc. The first closed-form solution for flow in a porous annulus when the amount of fluid entering through the outer wall is equal to the amount of fluid leaving the inner wall, was given by Berman [1]. Terril and Shreshtha [2] discussed this problem for the case when the walls are of different permeability. The problem of laminar non-Newtonian flow through a porous annulus has been discussed by Shreshtha [3]. Yeroshenko *et al.* [4] has solved the problem of heat transfer in laminar plane channel flow with uniform suction or injection by the method of separation of variables. Since perturbation techniques are impractical to solve the flow and heat transfer problems with a moderately large perturbation parameter, we have therefore adopted the quasilinearization technique in the present problem.

The method of quasilinearization has been employed by Huang [5, 6] in the problems of flow in a porous annulus for both Newtonian and non-Newtonian fluids. Many applications of the method of quasilinearization are found in other engineering fields [7, 8]. The advantages of this technique are: (i) the cross-flow Reynolds number can be chosen arbitrarily, (ii) the computational process converges quadratically [9], (iii) the basic equation can be used directly without much elaboration.

This note deals with the study of the heat transfer in the problem considered by Shreshtha [3] by the method of quasilinearization. The behaviour of the temperature profile has been studied for different sets of values of Reynolds number $R$, suction parameters $A$, $b$ and visco-inelastic number $\tau_2$ and is shown graphically.

## 2. FORMULATION OF THE PROBLEM

The constitutive equation of an incompressible Reiner–Rivlin fluid can be written as

$$\tau_{ij} = -p\delta_{ij}+2\mu_1 d_{ij}+4\mu_2 d_{i\alpha}d_j^{\alpha} \tag{1}$$

where

$$d_{ij} = \frac{1}{2}(v_{i,j}+v_{j,i}). \tag{2}$$

In a three-dimensional cylindrical set of coordinates $(r,\theta,z)$ the system consists of an annulus with porous walls of different permeability with the $z$-axis as the centre axis. The cross-flow velocity of suction at the inner wall ($U_a$) and that of injection at the outer wall ($U_b$) are constants. The inner wall ($r = a$) is maintained at constant temperature $T_1$ while the outer wall ($r = b$) at constant temperature $T_b$. A laminar steady motion of the non-Newtonian Reiner–Rivlin fluid through the annulus with velocity components $(u, v, w)$ in the positive directions of axes $(r,\theta,z)$ is considered. It is also assumed that $u$ is a function of $r$ alone and that fluid flow is swirl free and is then independent of $\theta$.

Using the constitutive equation (1) the equation of continuity, momentum equations and the energy equation can be written as

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z} = 0 \tag{3}$$

$$u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu_1\left(\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}-\frac{u}{r^2}\right)+4\nu_2\left\{\frac{\mathrm{d}u}{\mathrm{d}r}\left(2\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}+\frac{1}{2}\frac{\partial^2 w}{\partial r\partial z}\right)+\frac{1}{2}\frac{\partial w}{\partial r}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{2r}+\frac{\partial^2 w}{\partial z^2}\right)\right\}+2\nu_2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}-4\nu_2\frac{u^2}{r^3} \tag{4}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu_1\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right)+2\nu_2\frac{\mathrm{d}u}{\mathrm{d}r}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+4\nu_2\frac{\partial w}{\partial r}\left(\frac{1}{2}\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\frac{\partial^2 w}{\partial r\partial z}\right)+2\nu_2\frac{\partial w}{\partial z}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+4\frac{\partial^2 w}{\partial z^2}\right) \tag{5}$$

$$\rho c_v\left(u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\Phi \tag{6}$$

where

$$\Phi = \tilde{\tau}_j^i d_i^j. \tag{7}$$

The boundary conditions on the velocity profile and temperature are

$$\begin{aligned} u &= -U_a, \quad w = 0, \quad T = T_a \quad \text{at } r = a \\ u &= -U_b, \quad w = 0, \quad T = T_b \quad \text{at } r = b. \end{aligned} \tag{8}$$

For constant wall velocities, there exists a potential function $f(\xi)$, such that the radial component of velocity can be expressed as

$$u = U[f(\xi)/\xi^{1/2}]. \tag{9}$$

From the continuity equation (3) and expression (9) we get

$$w = \left[U_0-\frac{2z}{b}U\right]f'(\xi) \tag{10}$$

where the prime denotes differentiation with respect to $\xi$ and $U_0 f'(\xi)$ is the velocity profile of $w$ at $z = 0$ where cross-flow commences. Substituting expressions (9) and (10) into equations (4) and (5) and eliminating $p$ from the equations obtained we get

$$(\xi f^{\mathrm{iv}}+2f''')+R(f''f'-f'''f)-R\tau_2(f^{\mathrm{iv}}+2f'''f') = 0. \tag{11}$$

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## NOMENCLATURE

- $a$ radius of the inner wall of the annulus
- $A$ suction at the inner wall of the annulus, $\xi_0^{1/2} U_a/U$
- $b$ radius of the outer wall of the annulus
- $B$ suction at the outer wall of the annulus, $U_b/U$
- $c_v$ specific heat
- $d_{ij}$ strain rate tensor
- $d_j^\alpha$ mixed strain rate tensor
- $E$ Eckert number, $v_1 U/bc_v(T_b - T_a)$
- $k$ thermal conductivity
- $p$ hydrostatic pressure
- $P$ Prandtl number, $\rho v_1 c_v/k$
- $r$ radial coordinate
- $R$ cross-flow Reynolds number, $bU/2v_1$
- $T$ temperature
- $T_a$ temperature at the inner wall of the annulus
- $T_b$ temperature at the outer wall of the annulus
- $u$ radial velocity
- $U$ generalized cross-flow velocity, $|U_b| + (|U_a|a/b)$
- $U_0$ constant velocity parameter
- $U_a$ radial velocity at the inner wall of the annulus
- $U_b$ radial velocity at the outer wall of the annulus
- $v$ azimuthal velocity
- $v_{i,j}$ covariant derivative of the covariant velocity vector $v_i$
- $w$ axial velocity
- $z$ axial coordinate.

Greek symbols

- $\delta_{ij}$ Kronecker delta tensor
- $\zeta$ dimensionless axial variable, $\{(U_0/U) - (2z/b)\}$
- $\theta$ azimuthal coordinate
- $\mu_1$ Newtonian viscosity
- $\mu_2$ cross-viscosity
- $v_1$ kinematic Newtonian viscosity, $\mu_1/\rho$
- $v_2$ kinematic cross-viscosity, $\mu_2/\rho$
- $\xi$ dimensionless radial variable, $(r/b)^2$
- $\xi_0$ value of $\xi$ at the inner wall of the annulus, $(a/b)^2$
- $\rho$ density of the fluid
- $\tau_2$ visco-inelastic number, $4v_2/b^2$
- $\tau_{ij}$ stress tensor
- $\tilde{\tau}_j^i$ mixed deviatoric stress tensor
- $\phi$ dimensionless function of $\xi$
- $\Phi$ viscous dissipation function
- $\psi$ dimensionless function of $\xi$.

Equation (6) together with equations (9) and (10) suggest the form of the temperature distribution as follows:

$$T = T_a + \frac{v_1 U}{bc_v}\left[\phi(\xi) + \left(\frac{U_0}{U} - \frac{2z}{b}\right)^2 \psi(\xi)\right]. \tag{12}$$

Using equation (12) in equation (6), and equating the coefficient of $(U_0/U - 2z/b)^2$ and the terms independent of $(U_0/U - 2z/b)$ on both sides of the resulting equation, we obtain

$$\psi'' + \frac{1}{\xi}(1 - RPf)\psi' + \frac{2}{\xi}RPf'\psi + \frac{1}{\xi}RP(2\xi f''^2 - 3R\tau_2 f f''^2) = 0 \tag{13}$$

$$\phi'' + \frac{1}{\xi}(1 - RPf)\phi' + \frac{2}{\xi}\psi + \frac{2}{\xi^3}RP(4\xi^2 f'^2 - 2\xi f f' + f^2) + \frac{6}{\xi^3}\tau_2 R^2 P(-2\xi f f'^2 + f^2 f') = 0 \tag{14}$$

The expression of the temperature distribution in the dimensionless form can be expressed as

$$T^* = \frac{T - T_a}{T_b - T_a} = E(\phi + \zeta^2 \psi). \tag{15}$$

Boundary conditions (8) can be rewritten as

$$f(\xi_0) = -A, \quad f'(\xi_0) = 0, \quad \phi(\xi_0) = \psi(\xi_0) = 0$$

$$f(1) = -B, \quad f'(1) = 0, \quad \phi(1) = \frac{1}{E} = \omega \text{ (say)}, \quad \psi(1) = 0. \tag{16}$$

### 3. SOLUTION OF THE PROBLEM

To replace the differential equations (11), (13) and (14) by an equivalent system of first-order differential equations, we assume velocity and temperature functions and their derivatives as

$$(f, f', f'', f''') = (x_1, x_2, x_3, x_4)$$
$$(\psi, \psi', \phi, \phi') = (y_1, y_2, y_3, y_4). \tag{17}$$

Using the technique of quasilinearization as given in refs. [5, 6, 17] the non-linear differential equations (11), (13) and (14) can be replaced by the equivalent system of first-order linearized differential equations as follows:

$$\frac{dx_1^{(k+1)}}{d\xi} = x_2^{(k+1)}; \quad \frac{dx_2^{(k+1)}}{d\xi} = x_3^{(k+1)}; \quad \frac{dx_3^{(k+1)}}{d\xi} = x_4^{(k+1)};$$

$$\frac{dx_4^{(k+1)}}{d\xi} = \frac{1}{(\xi - \tau_2 R x_1^{(k)})}[R x_4^{(k)} x_1^{(k+1)} - R(x_3^{(k)} - 2\tau_2 x_4^{(k)}) x_2^{(k+1)} - R x_2^{(k)} x_3^{(k+1)} - (2 - R x_1^{(k)} - 2\tau_2 R x_2^{(k)}) x_4^{(k+1)}] + \frac{R}{(\xi - \tau_2 R x_1^{(k)})} \times [x_2^{(k)} x_3^{(k)} - x_1^{(k)} x_4^{(k)} - 2\tau_2 x_2^{(k)} x_4^{(k)}] \tag{18}$$

$$\frac{dy_1^{(k+1)}}{d\xi} = y_2^{(k+1)};$$

$$\frac{dy_2^{(k+1)}}{d\xi} = \frac{1}{\xi}[-y_2^{(k+1)} + RP(x_1^{(k+1)} y_2^{(k+1)} - 2x_2^{(k+1)} y_1^{(k+1)}) + 3R^2 P\tau_2 x_1^{(k+1)} (x_3^{(k+1)})^2] - 2RP(x_3^{(k+1)})^2;$$

$$\frac{dy_3^{(k+1)}}{d\xi} = y_4^{(k+1)};$$

$$\frac{dy_4^{(k+1)}}{d\xi} = \frac{1}{\xi}[(-1 + RPx_1^{(k+1)}) y_4^{(k+1)} - 2y_1^{(k+1)} - 8RP(x_2^{(k+1)})^2] + \frac{4}{\xi^2}RP[x_1^{(k+1)} x_2^{(k+1)} + 3R\tau_2 x_1^{(k+1)} (x_2^{(k+1)})^2] - \frac{2RP}{\xi^3}[(x_1^{(k+1)})^2 + 3R\tau_2 (x_1^{(k+1)})^2 x_2^{(k+1)}] \tag{19}$$

where $x_i^{(k+1)}$ and $y_i^{(k+1)}$ for $i = 1, 2, 3, 4$ are the $(k+1)$th approximations to the solutions of the differential equations (11), (13) and (14). The $k$th approximations $x_i^{(k)}$ and $y_i^{(k)}$ are assumed to be a set of known approximate solutions.

Boundary conditions (16) become

$$x_1^{(k+1)}(\xi_0) = -A, \quad x_2^{(k+1)}(\xi_0) = 0, \quad y_1^{(k+1)}(\xi_0) = 0,$$
$$y_3^{(k+1)}(\xi_0) = 0, \quad x_1^{(k+1)}(1) = -B, \quad x_2^{(k+1)}(1) = 0,$$
$$y_1^{(k+1)}(1) = 0, \quad y_3^{(k+1)}(1) = \omega. \tag{20}$$

The system of equations (18) and (19) is linear in $x_i^{(k+1)}$ and $y_i^{(k+1)}$, respectively, thus the principle of superposition can be used to obtain the general solution. The initial conditions for $x_i^{(k+1)}$ and $y_i^{(k+1)}$, respectively, are chosen as

$$P_i^{(k+1)}(\xi_0) = (-A,0,0,0), \quad G_{i3}^{(k+1)}(\xi_0) = (0,0,1,0), \quad G_{i4}^{(k+1)}(\xi_0) = (0,0,0,1) \tag{21}$$

and

$$Q_i^{(k+1)}(\xi_0) = (0,0,0,0), \quad H_{i2}(\xi_0) = (0,1,0,0), \quad H_{i4}^{(k+1)}(\xi_0) = (0,0,0,1). \tag{22}$$

In initial conditions (21) and (22) $P_i^{(k+1)}(\xi_0)$ and $Q_i^{(k+1)}(\xi_0)$ are taken for the initial values of the particular solutions of equations (18) and (19) while $G_{i3}^{(k+1)}(\xi_0)$, $G_{i4}^{(k+1)}(\xi_0)$ and $H_{i2}^{(k+1)}(\xi_0)$, $H_{i4}^{(k+1)}(\xi_0)$ are used for their homogeneous solutions, respectively.

The general solutions of equations (18) and (19) can be obtained as

$$x_i^{(k+1)}(\xi) = P_i^{(k+1)}(\xi)+C_3^{(k+1)}G_{i3}^{(k+1)}(\xi) + C_4^{(k+1)}G_{i4}^{(k+1)}(\xi) \tag{23}$$

and

$$y_i^{(k+1)}(\xi) = Q_i^{(k+1)}(\xi)+D_2^{(k+1)}H_{i2}^{(k+1)}(\xi)+D_4^{(k+1)}H_{i4}^{(k+1)}(\xi) \tag{24}$$

where $C_3^{(k+1)}$, $C_4^{(k+1)}$, $D_2^{(k+1)}$ and $D_4^{(k+1)}$ are unknown constants which can be determined by using boundary conditions (20) at $\xi = 1$.

## 4. RESULTS, DISCUSSIONS AND CONCLUSIONS

The system of equations (18) and (19) is solved numerically by the fourth-order Runge–Kutta method. The values of the functions $f, f'$, etc., $\psi$, $\phi$, $\psi'$ and $\phi'$ have been computed for different values of $R$, $A$, $B$ and $\tau_2$ correct to six decimal places. It is found that $f, f'$, etc. are in good agreement with their values obtained by Huang [6]. Hence, here we are confined only to the study concerned with the behaviour of the temperature profile for different values of $R$, $A$, $B$ and $\tau_2$. Without loss of generality, the permeability at any wall of the annulus is chosen to be negative (i.e. $A, B > 0$) with

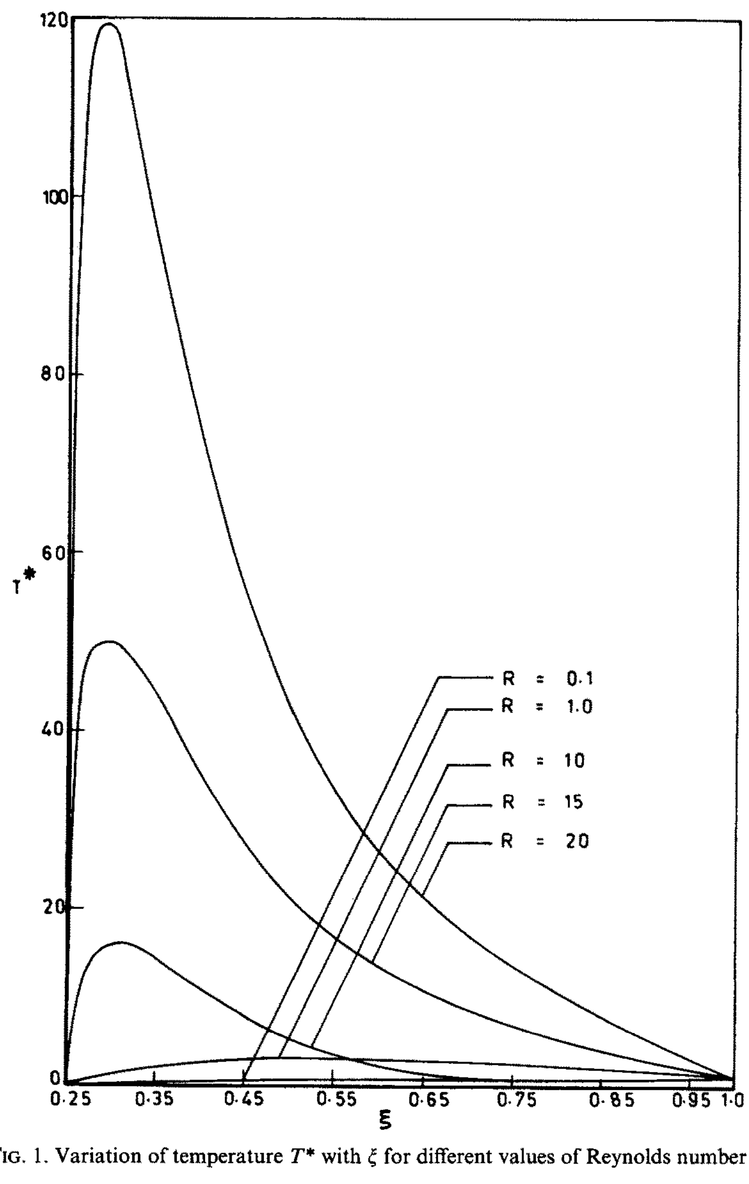


FIG. 1. Variation of temperature $T^*$ with $\xi$ for different values of Reynolds number $R$.

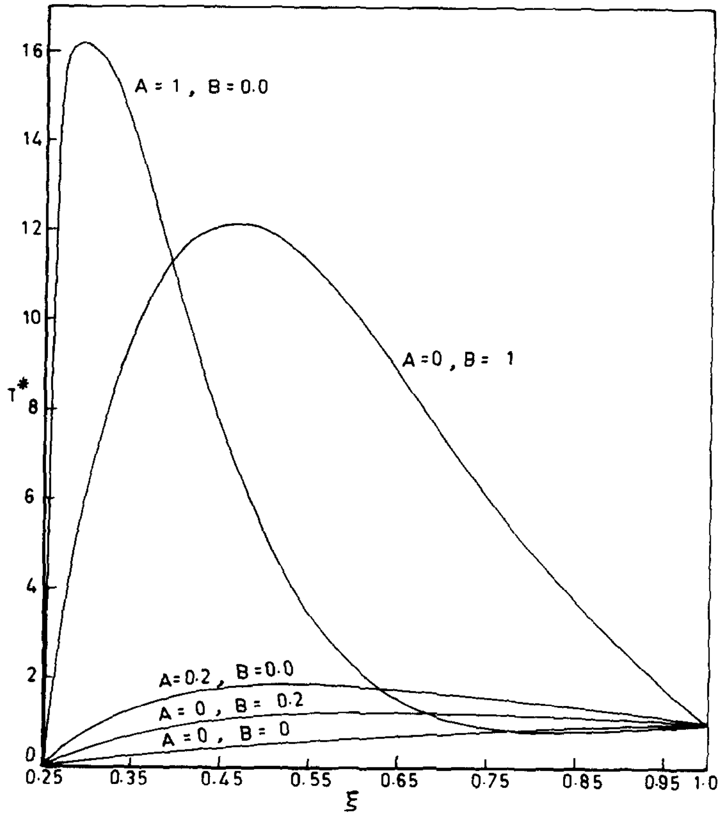


FIG. 2. Variation of temperature $T^*$ with $\xi$ for different values of suction parameters $A$ and $B$.

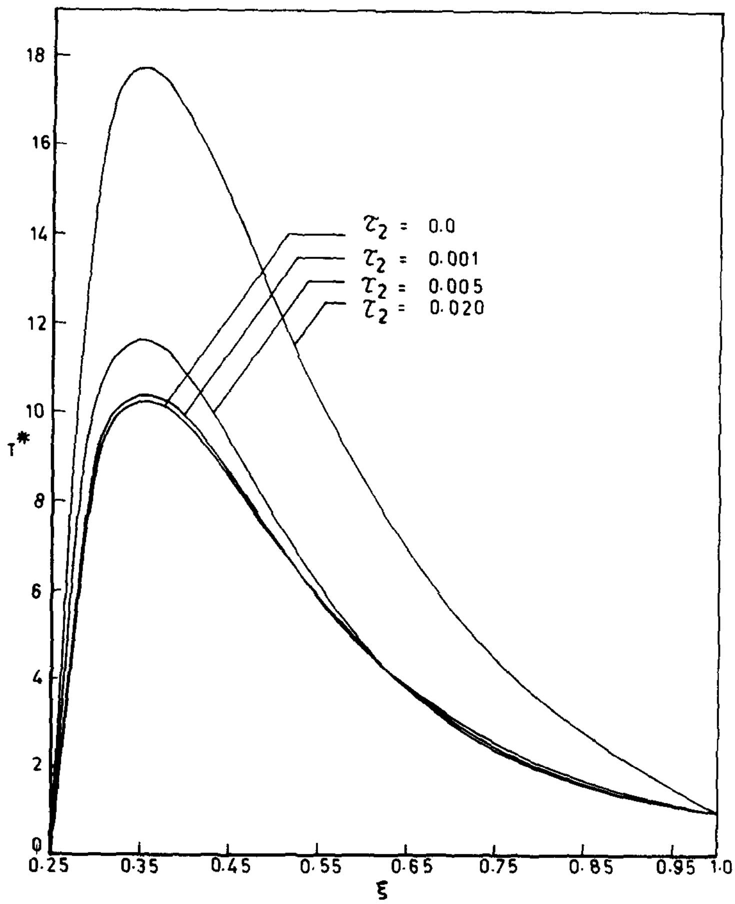


FIG. 3. Variation of temperature $T^*$ with $\xi$ for different values of visco-inelastic number $\tau_2$.

another wall of the annulus impermeable (i.e. $A = 0$ if it is inner wall and $B = 0$ if it is outer wall of the annulus). It means there exists a suction cross flow either through the inner wall or injection through the outer wall of the annulus.

The convergence rate of the quasilinearization method is fairly fast, only a few iterations are needed to obtain a five digit accuracy. The method gives quite accurate results for moderately large values of Reynolds number up to the desired accuracy and hence may be very useful in solving two-point boundary value problems in the case of non-Newtonian fluids.

The variation of the temperature profile at $A = 1.0$, $B = 0$, $\tau_2 = 0.02$, $\zeta = 0.4$, $P = 0.4$, $E = 1$ for $R = 0.1, 1, 10, 15, 20$ is represented in Fig. 1. It is evident that for small Reynolds number, temperature increases linearly with $\xi$ throughout the annulus. In the case of moderately large Reynolds number, the temperature increases very rapidly at first and then starts decreasing rapidly thereafter. It is also seen that an increase in $R$ increases the temperature very rapidly near the inner wall of the annulus while increases very slowly near the outer wall of the annulus. Hence there is a heating near the inner wall of the annulus which increases with an increase in $R$.

Figure 2 shows the variation of the temperature profile at $R = 10$, $\tau_2 = 0.02$, $\zeta = 0.4$, $P = 0.4$, $E = 1$ for $A = 0, 0.2, 1.0$ when $B = 0$ and for $B = 0, 0.2, 0.1$ when $A = 0$. In case of $A = B = 0$ the temperature increases with an increase in $\xi$ throughout the annulus. Though in both the cases $B = 0$ and $A = 0, 0.2, 1$; $A = 0$ and $B = 0, 0.2, 1.0$ the temperature increases rapidly near the inner wall of the annulus but the rate of increment with $A$ is much faster than that with $B$. In other words there is more heating near the inner wall in case of suction on the inner wall than that for the injection at the outer wall of the annulus.

Variation of the temperature profile at $R = 10$, $A = 1$, $B = 0$, $\zeta = 0.2$, $P = 0.4$, $E = 1$ for $\tau_2 = 0, 0.001, 0.005, 0.02$ is shown in Fig. 3. It is clear from the figure that the temperature increases rapidly with an increase in $\tau_2$ near the inner wall of the annulus and decreases for $\tau_2 = 0.001$ and 0.005 to that for $\tau_2 = 0$ thereafter. It is also evident that there is more heating near the inner wall in case of Reiner–Rivlin fluid than that for a Newtonian fluid.

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# The general characteristics of boiling heat transfer from a surface embedded in a porous medium

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## REVIEW OF THE LITERATURE

IN 1985, Afgan *et al.* [1] conducted experiments for nucleate boiling of water, ethyl alcohol and Freon-113 using horizontal tubes covered with porous layers consisting of scintered dendrite shaped and spherical 63–100 $\mu$m particles. In one set of their experiments, the heat flux $q$ vs the degree of superheat or 'excess temperature', $\Delta T_x$, was obtained for a 16 mm diameter electrically heated tube covered by a 2.2 mm thick porous layer saturated with water. In this experiment, they observed that the boiling process can be divided into three regimes. In the first regime, called mode I, which corresponds to relatively small $\Delta T_x$, normal bubble boiling takes place. The second regime, called the transition regime, occurs at higher $\Delta T_x$. The investigators observed that, in the transition regime, the temperature difference increased by a large increment for a constant value of heat flux after a sufficiently long time interval. They attributed this phenomenon to the slow growth of a vapor film through the porous layer.

After the transition regime, a third regime, called mode II, starts. In this regime $\Delta T_x$ increases with increasing heat flux, there is a vapor film inside the porous layer and nucleate boiling occurs at the outer surface of the porous layer. The transition from mode I to mode II can follow along different paths if the time necessary to achieve steady state is not allowed. Mode II persists until a boiling crisis or 'burn out' point is reached.